# CHAPTER 17

# Hyperbola

- Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2 = a^2(e^2 - 1)$

or $a^2 e^2 = a^2 + b^2$ *i.e.* $e^2 = 1 + \frac{b^2}{a^2}=1+\left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

**(*a*) Foci:**

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

**(*b*) Equations of Directrices:**

$x=\frac{a}{e}$ & $x=-\frac{a}{e}$.

**(*c*) Vertices:**

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

**(*d*) Latus Rectum:**

(*i*) Equation: $x = \pm ae$

(*ii*) Length $=\frac{2b^2}{a}=\frac{(\text{Conjugate Axis})}{(\text{Transverse Axis})}=2a(e^2-1)$

$= 2e$(distance from focus to directrix)

(*iii*) Ends: $\left(ae,\frac{b^2}{a}\right),\left(ae,\frac{-b^2}{a}\right);\left(-ae,\frac{b^2}{a}\right),\left(-ae,\frac{-b^2}{a}\right)$

**(*e*) Focal Property:**

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis *i.e.* $||PS| - |PS'|| = 2a$. The distance $SS'$ = focal length.

**(*f*) Focal Distance:**

Distance of any point $P(x, y)$ on hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

**Conjugate Hyperbola:** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ & $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are conjugate hyperbolas of each.

**Auxillary Circle:** $x^2 + y^2 = a^2$.

**Parametric Representation:** $x = a\sec\theta$ & $y = b\tan\theta$

**Position of *A* point '*P*' w.r.t. *A* Hyperbola:**

$S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1 >=$ or $<0$ according as the point $(x_1, y_1)$ lines inside, on

or outside the curve.

## Tangents

(*i*) **Slope Form:** $y = m \times \pm\sqrt{a^2m^2-b^2}$

(*ii*) **Point Form:** at the point $(x_1, y_1)$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(*iii*) **Parametric Form:** $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

- **Normal to The Hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(*a*) **Point form:** Equation of the normal to the given hyperbola at the point $P(x_1, y_1)$ on it is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2=a^2e^2$.

(*b*) **Slope form:** The equation of normal of slope $m$ to the given hyperbola is $y=mx\mp\frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$ foot of normal are $\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}},\mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$.

(*c*) **Parametric form:** The equation of the normal at the point P $(a\sec\theta, b\tan\theta)$ to the given hyperbola is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2$.

## Director Circle

Equation to the director circle is: $x^2 + y^2 = a^2 - b^2$.

## Chord of Contact

If $PA$ and $PB$ be the tangents from point $P(x_1, y_1)$ to the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then the equation of the chord of contact $AB$ is

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T = 0$ at $(x_1, y_1)$.

## Equation of Chord with mid Point $(x_1, y_1)$

The equation of the chord of the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, whose mid-point be $(x_1, y_1)$ is $T = S_1$ where $T$

$=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1,\ S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)$.

## Asymptotes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

**Reflection property of the hyperbola:** An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

**Rectangular or Equilateral Hyperbola:** $xy = c^2$, eccentricity is $\sqrt{2}$.

**Vertices:** $(\pm c \pm c)$ ; Focii : $\left(\pm\sqrt{2}c, \pm\sqrt{2}c\right)$. Directrices : $x + y = \pm\sqrt{2}\,c$.

**Latus Rectum ($l$):** $= l = 2\sqrt{2}\,c = T.A. = C.A.$

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$ & at $P(t)$ is $\frac{x}{t}+ty=2c$.

Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

Chord with a given middle point as $(h, k)$ is $kx + hy = 2hk$.